

# CROSSED POINTED CATEGORIES AND THEIR EQUIVARIANTIZATIONS

DEEPAK NAIDU

**ABSTRACT.** We propose the notion of *quasi-abelian third cohomology* of crossed modules, generalizing Eilenberg and MacLane's abelian cohomology and Ospel's quasi-abelian cohomology, and classify crossed pointed categories in terms of it. We apply the process of equivariantization to the latter to obtain braided fusion categories which may be viewed as generalizations of the categories of modules over twisted Drinfeld doubles of finite groups. As a consequence, we obtain a description of *all* braided group-theoretical categories. A criterion for these categories to be modular is given. We also describe the quasi-triangular quasi-Hopf algebras underlying these categories.

## 1. INTRODUCTION

The notion of a crossed category (short for braided group-crossed category), introduced by Turaev [Tu1, Tu2], has attracted much attention recently [DGNO, Ki1, Ki2, M1, M2]. Roughly, a crossed category consists of a group  $G$ , a  $G$ -graded tensor category  $\mathcal{C}$ , an action  $g \mapsto T_g$  of  $G$  on  $\mathcal{C}$  by tensor autoequivalences, and  $G$ -braidings  $c(X, Y) : X \otimes Y \xrightarrow{\sim} T_g(Y) \otimes X$ ,  $X, Y \in \mathcal{C}$ , satisfying certain compatibility conditions. Crossed categories are known to arise in various contexts; for instance, in [M1], Müger showed that Galois extensions of braided tensor categories have a natural structure of crossed categories. Müger also established a connection between 1-dimensional quantum field theories and crossed categories [M2]. Furthermore, crossed categories have been shown, by Kirillov Jr., also to arise in the theory of vertex operator algebras [Ki2].

Recall that a fusion category is said to be *pointed* if all its simple objects are invertible. One of our goals in the present note is to classify all crossed pointed categories. It is known [JS] that braided pointed categories are classified by Eilenberg and MacLane's abelian cohomology  $H_{ab}^3(A, \mathbb{K}^\times)$ , where  $A$  is a finite abelian group. On the other hand, in [Tu1, Tu2], a description of certain crossed pointed categories in which the group action is strict is given in terms of Ospel's quasi-abelian cohomology  $H_{qa}^3(G, \mathbb{K}^\times)$ , where  $G$  is a (not necessarily abelian) finite group. As remarked in [M2, Subsection 4.9], to obtain a complete classification of crossed pointed categories one must allow for non-strict group actions. To this end, we generalize Ospel's quasi-abelian cohomology to define the notion of quasi-abelian third cohomology  $H_{qa}^3(\mathcal{X}, \mathbb{K}^\times)$  of a crossed module  $\mathcal{X}$  (see Definition 3.4). To any given  $\xi \in Z_{qa}^3(\mathcal{X}, \mathbb{K}^\times)$  we associate a crossed pointed category  $\mathcal{C}(\xi)$  and show that all crossed pointed categories are of this form.

Another notion that has been studied extensively recently is that of a modular category. Examples of modular categories arise from several diverse areas such as quantum group theory, 3-dimensional topology, vertex operator algebras, rational conformal field

theory, etc. Let  $G$  be a finite group. Perhaps the most accessible construction of a modular category is that of the category of modules over the Drinfeld double  $D(G)$  of  $G$ . Let  $\omega$  be a 3-cocycle on  $G$ . In [DPR1, DPR2] Dijkgraaf, Pasquier, and Roche introduced a quasi-triangular quasi-Hopf algebra  $D^\omega(G)$ , generalizing the Drinfeld double  $D(G)$ . It is well known that the category  $D^\omega(G)\text{-Mod}$  of modules over  $D^\omega(G)$  is a modular category. Modular categories resembling  $D^\omega(G)\text{-Mod}$  arises naturally from crossed pointed categories. Specifically, an important feature of a general crossed fusion category is that the application of the equivariantization process (which is analogous to taking the invariants under a groups action) yields a braided fusion category. We apply the equivariantization process to the aforementioned crossed pointed category  $\mathcal{C}(\xi)$  and study the resulting braided fusion category, which resembles the category  $D^\omega(G)\text{-Mod}$ . As a consequence, we obtain a description of *all* braided group-theoretical categories. We show that  $\mathcal{C}(\xi)$  is modular if and only if  $\xi$  is nondegenerate in the sense of Definition 3.10 and a certain homomorphism is surjective (see Proposition 5.6).

By a general result, the equivariantization of the category  $\mathcal{C}(\xi)$  is equivalent, as a braided fusion category, to the category of modules over some finite-dimensional quasi-triangular quasi-Hopf algebra  $H$ . In the sequel we describe such an  $H$ . Namely, given  $\xi \in Z_{qa}^3(\mathcal{X}, \mathbb{K}^\times)$ , we construct a finite-dimensional quasi-triangular quasi-Hopf algebra  $H(\xi)$ , generalizing  $D^\omega(G)$ , and show that  $\mathcal{C}(\xi) \cong H(\xi)\text{-Mod}$ , as braided fusion categories.

The content of this note is as follows. In Section 2, we recall some essential definitions and results concerning nondegenerate fusion categories, equivariantization, and crossed categories. In Section 3, we propose the notion of quasi-abelian third cohomology of crossed modules. Section 4 consists of a construction of crossed pointed categories from quasi-abelian 3-cocycles and a classification of the former. In Section 5, we apply the process of equivariantization to the categories obtained in Section 4 and study the resulting braided fusion categories. In Section 6, we construct finite-dimensional quasi-triangular quasi-Hopf algebras from quasi-abelian 3-cocycles, which are shown to underlie the braided fusion categories obtained in the Section 5.

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## 2. PRELIMINARIES

In this note, we will freely use the language and basic theory of fusion categories and modular categories [BK, Os, ENO]. In what follows we recall some essential definitions and results.

**2.1. Convention.** Let  $\mathbb{K}$  be an algebraically closed field of characteristic 0. The multiplicative group of non-zero elements of  $\mathbb{K}$  will be denoted by  $\mathbb{K}^\times$ . Unless otherwise stated, all cocycles appearing in this work will have coefficients in the trivial module  $\mathbb{K}^\times$ . All functors will be assumed to be additive and  $\mathbb{K}$ -linear on the morphism spaces. The unit object of a tensor category will be denoted by  $\mathbb{1}$ . The identity element of a group will be denoted by  $e$ .

**2.2. Morita equivalence.** Following [M4], we say that two fusion categories  $\mathcal{C}$  and  $\mathcal{D}$  are Morita equivalent if  $\mathcal{D}$  is equivalent to the dual fusion category  $\mathcal{C}_{\mathcal{M}}^*$ , for some indecomposable right  $\mathcal{C}$ -module category  $\mathcal{M}$  (see also [ENO, O]). The above is known

to be an equivalence relation on the class of fusion categories. A fusion category is said to be *pointed* if all its simple objects are invertible. A fusion category is *group-theoretical* if it is Morita equivalent to a pointed category.

**2.3. Nondegenerate fusion categories.** Let  $\mathcal{C}$  be a braided fusion category with braiding  $c$ . Two objects  $X$  and  $Y$  of  $\mathcal{C}$  are said to *centralize* each other if  $c(Y, X) \circ c(X, Y) = \text{id}_{X \otimes Y}$  [M3].

For any fusion subcategory  $\mathcal{D} \subseteq \mathcal{C}$  its *centralizer*  $\mathcal{D}'$  is the full fusion subcategory of  $\mathcal{C}$  consisting of all objects  $X \in \mathcal{C}$  which centralize every object in  $\mathcal{D}$ . The category  $\mathcal{C}$  is said to be *non-degenerate* if  $\mathcal{C}' = \text{Vec}$  (the fusion category generated by the unit object). If  $\mathcal{C}$  is a pre-modular category, i.e., has a twist, then it is non-degenerate if and only if it is modular [BB, M3, DGNO].

The following proposition will be used later.

**Proposition 2.1.** *Let  $\mathcal{C}$  be a nondegenerate fusion category. Suppose  $\mathcal{C}$  admits a twist. Then the set of twists on  $\mathcal{C}$  is in bijection with the set of invertible self-dual objects of  $\mathcal{C}$ .*

*Proof.* Let  $\text{Aut}_{\otimes}(\text{id}_{\mathcal{C}})$  denote the group of tensor automorphisms of the identity tensor functor  $\text{id}_{\mathcal{C}}$ . Define  $\text{Aut}_{\otimes}^*(\text{id}_{\mathcal{C}}) := \{\varphi \in \text{Aut}_{\otimes}(\text{id}_{\mathcal{C}}) \mid \varphi_{X^*} = (\varphi_X)^*, \forall X \in \mathcal{C}\}$ . Let  $\theta$  be a fixed twist on  $\mathcal{C}$ . The map  $\varphi \mapsto \theta_{\varphi}, (\theta_{\varphi})_X := \theta_X \circ \varphi_X$  for all  $X \in \mathcal{C}$ , is a bijection from  $\text{Aut}_{\otimes}^*(\text{id}_{\mathcal{C}})$  to the set of all twists on  $\mathcal{C}$ .

Let  $X_1, X_2, \dots$  denote the simple objects of  $\mathcal{C}$  and let  $G(\mathcal{C})$  denote the group of invertible objects of  $\mathcal{C}$ . Also, let  $S$  denote the  $S$ -matrix of  $\mathcal{C}$  with respect to  $\theta$ . It was shown in [GN] that the map

$$G(\mathcal{C}) \rightarrow \text{Aut}_{\otimes}(\text{id}_{\mathcal{C}}) : X_j \mapsto \varphi_j, \quad (\varphi_j)_{X_i} := \frac{S_{ij}}{d(X_i)d(X_j)} \text{id}_{X_i}$$

is an isomorphism. It is easy to check that this map restricts to a bijection between the set of invertible self-dual object of  $\mathcal{C}$  and the set  $\text{Aut}_{\otimes}^*(\text{id}_{\mathcal{C}})$ .  $\square$

**2.4. Equivariantization.** Recall that a tensor functor between two tensor categories  $\mathcal{C}$  and  $\mathcal{D}$  is a triple  $(F, \varphi, \varphi_0)$  where  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a functor,  $\varphi$  is a natural isomorphism  $F \circ \otimes_{\mathcal{C}} \xrightarrow{\sim} \otimes_{\mathcal{D}} \circ (F \times F)$ , and  $\varphi_0$  is an isomorphism  $F(\mathbb{1}_{\mathcal{C}}) \xrightarrow{\sim} \mathbb{1}_{\mathcal{D}}$  satisfying certain compatibility conditions (see [K]). We will call  $\varphi$  the *tensor structure* on  $F$  and  $\varphi_0$  the *unit-preserving structure* on  $F$ . For a group  $G$ , we will denote by  $\underline{G}$  the tensor category whose objects are elements of  $G$ , morphisms are the identities, and whose tensor product is given by the group operation in  $G$ .

Let  $\mathcal{C}$  be a fusion category with an action of a finite group  $G$  given by a tensor functor  $T : \underline{G} \rightarrow \text{Aut}_{\otimes}(\mathcal{C}); g \mapsto T_g$ . Let  $\gamma$  be the tensor structure on the functor  $T$ . In this situation one can define the notion of a  $G$ -equivariant object in  $\mathcal{C}$ . Namely, a  $G$ -equivariant object in  $\mathcal{C}$  is a pair  $(X, \{u_g\}_{g \in G})$  where  $X$  is an object of  $\mathcal{C}$  and

$$(1) \quad u_g : T_g(X) \xrightarrow{\sim} X, \quad g \in G,$$

is a family of isomorphisms called *equivariant structure on  $X$*  such that

$$(2) \quad u_{gh} = u_g \circ T_g(u_h) \circ \gamma_{g,h}(X),$$

for all  $g, h \in G$ .

One defines morphisms between equivariant objects to be morphisms in  $\mathcal{C}$  commuting with the equivariant structures. The *equivariantization of  $\mathcal{C}$* , denoted  $\mathcal{C}^G$ , is the category of  $G$ -equivariant objects of  $\mathcal{C}$  [Kil, AG, G, Ta]. The equivariantization category  $\mathcal{C}^G$  is a fusion category with tensor product given by the following. Let  $(X, \{u_g\}_{g \in G}), (X', \{u'_g\}_{g \in G}) \in \mathcal{C}^G$ . Then

$$(X, \{u_g\}_{g \in G}) \otimes (X', \{u'_g\}_{g \in G}) := (X \otimes X', \{\tilde{u}_g\}_{g \in G}),$$

where

$$(3) \quad \tilde{u}_g := (u_g \otimes u'_g) \circ \mu_g(X, X')$$

for all  $g \in G$ . Here  $\mu_g$  is the tensor structure on the functor  $T_g, g \in G$ .

**Remark 2.2.** We have  $\text{FPdim}(\mathcal{C}^G) = |G| \text{FPdim}(\mathcal{C})$ .

**2.5. Crossed categories.** Recall that a *grading* of a fusion category  $\mathcal{C}$  by a finite group  $G$  is a decomposition

$$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$$

of  $\mathcal{C}$  into a direct sum of full abelian subcategories such that  $\otimes$  maps  $\mathcal{C}_g \times \mathcal{C}_h$  to  $\mathcal{C}_{gh}$  and  $*$  maps  $\mathcal{C}_g$  to  $\mathcal{C}_{g^{-1}}$ , for all  $g, h \in G$ . Note that  $\mathcal{C}_e$ , called the *trivial component*, is a fusion subcategory of  $\mathcal{C}$ . A grading is said to *faithful* if  $\mathcal{C}_g \neq 0$  for all  $g \in G$ .

Below we recall the notion of a *crossed category* (short for *braided group-crossed category*), introduced by Turaev [Tu1, Tu2], in a more general form (see also [DGNO, M1, M2]).

**Definition 2.3.** A *crossed fusion category* is an eight-tuple  $(\mathcal{C}, G, T, \gamma, \iota, \mu, \nu, c)$ , where

- $G$  is a finite group,
- $\mathcal{C}$  is a fusion category with a (not necessarily faithful)  $G$ -grading  $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ ,
- $T : \underline{G} \rightarrow \text{Aut}_{\otimes}(\mathcal{C}) : g \mapsto T_g$  is a tensor functor, satisfying  $T_g(\mathcal{C}_h) \subset \mathcal{C}_{ghg^{-1}}$ , with tensor structure  $\gamma$  and unit-preserving structure  $\iota$ ,
- $\mu$  is a family  $\{\mu_g\}_{g \in G}$  where  $\mu_g$  is a tensor structure on  $T_g$ ,
- $\nu$  is a family  $\{\nu_g\}_{g \in G}$  where  $\nu_g$  is a unit-preserving structure on  $T_g$ ,
- $c(X, Y) : X \otimes Y \xrightarrow{\sim} T_g(Y) \otimes X, X \in \mathcal{C}_g, Y \in \mathcal{C}$ , is a family of natural isomorphisms, called  *$G$ -braiding*,

satisfying the following compatibility conditions:

$$(i) \quad (\gamma_{g,h}(Y) \otimes \text{id}_{T_g(X)}) \circ (\gamma_{ghg^{-1},g}^{-1}(Y) \otimes \text{id}_{T_g(X)}) \circ c(T_g(X), T_g(Y)) \circ \mu_g(X, Y) \\ = \mu_g(T_h(Y), X) \circ T_g(c(X, Y)),$$

for all  $g, h \in G$  and objects  $X \in \mathcal{C}_h, Y \in \mathcal{C}$ .

$$(ii) \quad \alpha_{T_g(T_h(Z)), X, Y}^{-1} (\gamma_{g,h}(Z) \otimes \text{id}_{X \otimes Y}) \circ c(X \otimes Y, Z) \circ \alpha_{X, Y, Z}^{-1} \\ = (c(X, T_h(Z)) \otimes \text{id}_Y) \circ \alpha_{X, T_h(Z), Y}^{-1} \circ (\text{id}_X \otimes c(Y, Z)),$$

for all  $g, h \in G$  and objects  $X \in \mathcal{C}_g, Y \in \mathcal{C}_h, Z \in \mathcal{C}$ .

$$(iii) \quad \alpha_{T_g(Y), T_g(Z), X} \circ (\mu_g(Y, Z) \otimes \text{id}_X) \circ c(X, Y \otimes Z) \circ \alpha_{X, Y, Z} \\ = (\text{id}_{T_g(Y)} \otimes c(X, Z)) \circ \alpha_{T_g(Y), X, Z} \circ (c(X, Y) \otimes \text{id}_Z),$$

for all  $g \in G$  and objects  $X \in \mathcal{C}_g, Y, Z \in \mathcal{C}$ .

(Here  $\alpha$  denotes the associativity constraint of  $\mathcal{C}$ .)

**Remark 2.4.** The trivial component of a crossed fusion category is a braided fusion category.

Now let  $\mathcal{C} := (\mathcal{C}, G, T, \gamma, \iota, \mu, \nu, c)$  be a crossed fusion category. It is explained in [Kil] and [M1] that the equivariantization category  $\mathcal{C}^G$  admits a braiding, i.e.  $\mathcal{C}^G$  is a braided fusion category. The braiding  $\tilde{c}$  on  $\mathcal{C}^G$  is defined as follows. Let  $(X, \{u_g\}_{g \in G})$  and  $(X', \{u'_g\}_{g \in G})$  be objects of  $\mathcal{C}^G$ . Let  $X = \bigoplus_{g \in G} X_g$  be a decomposition of  $X$  with respect to the  $G$ -grading of  $\mathcal{C}$ . Then  $\tilde{c}_{X, X'}$  is given by the composition

$$(4) \quad X \otimes X' = \bigoplus_{g \in G} X_g \otimes X' \xrightarrow{\oplus c_{X_g, X'}} \bigoplus_{g \in G} T_g(X') \otimes X_g \xrightarrow{\oplus u'_g \otimes \text{id}_{X_g}} \bigoplus_{g \in G} X' \otimes X_g = X' \otimes X.$$

**Remark 2.5.** It is shown in [DGNO] that the equivariantization category  $\mathcal{C}^G$  is non-degenerate if and only if the  $G$ -grading is faithful and the trivial component  $\mathcal{C}_e$  is nondegenerate.

**Definition 2.6.** Let  $\mathcal{C} = (\mathcal{C}, G, T, \gamma, \iota, \mu, \nu, c)$  and  $\mathcal{C}' = (\mathcal{C}', G', T', \gamma', \iota', \mu', \nu', c')$  be crossed fusion categories. A *crossed tensor functor* from  $\mathcal{C}$  to  $\mathcal{C}'$  is a 5-tuple  $(f, F, \eta, \eta_0, \beta)$  where

- $f : G \rightarrow G'$  is a group homomorphism,
- $F : \mathcal{C} \rightarrow \mathcal{C}'$  is a tensor functor with tensor structure  $\eta$  and unit-preserving structure  $\eta_0$ ,
- $\beta$  is a family  $\{\beta_g\}_{g \in G}$  where  $\beta_g : F \circ T_g \xrightarrow{\sim} T'_{f(g)} \circ F$  is an isomorphism of tensor functors

satisfying the following compatibility conditions:

$$(i) \quad F(\mathcal{C}_g) \subseteq \mathcal{C}'_{f(g)},$$

for all  $g \in G$ .

$$(ii) \quad (\beta_g(Y) \otimes \text{id}_{F(X)}) \circ \eta(T_g(Y), X) \circ F(c(X, Y)) = c'(F(X), F(Y)) \circ \eta(X, Y),$$

for all  $g \in G$  and objects  $X \in \mathcal{C}_g, Y \in \mathcal{C}$ .

$$(iii) \quad T'_{f(g)}(\beta_h(X)) \circ \beta_g(T_h(X)) \circ F(\gamma_{g,h}(X)) = \gamma'_{f(g), f(h)}(F(X)) \circ \beta_{gh}(X),$$

for all  $g, h \in G$  and objects  $X \in \mathcal{C}$ .

We say that  $(f, F, \eta, \eta_0, \beta)$  is an *equivalence* if  $f$  is an isomorphism and  $F$  is an equivalence.

**2.6. Pointed Categories.** Recall that a fusion category is said to be *pointed* if all its simple object are invertible.

Let  $X$  be a finite group and  $\omega$  be a 3-cocycle on  $X$ . We associate to the pair  $(X, \omega)$  a pointed category  $\text{Vec}_X^\omega$  as follows. The objects of  $\text{Vec}_X^\omega$  are  $X$ -graded finite dimensional vector spaces over  $\mathbb{K}$  and morphisms are linear transformations that respect the grading. The unit object of  $\text{Vec}_X^\omega$  is ground field  $\mathbb{K}$  supported on  $\{e\}$ . The tensor product  $V \otimes W$  of homogeneous objects  $V, W \in \text{Vec}_X^\omega$  of degrees  $x, y \in X$ , respectively, is defined to be the homogeneous object  $V \otimes_{\mathbb{K}} W$  of degree  $xy$ .

The associativity constraint  $\alpha$  is defined by

$$\alpha_{U,V,W} : (U \otimes V) \otimes W \xrightarrow{\sim} U \otimes (V \otimes W) : (u \otimes v) \otimes w \mapsto \omega(x, y, z)u \otimes (v \otimes w),$$

where  $U, V, W \in \text{Vec}_G^\omega$  and  $u \in U, v \in V, w \in W$  are homogeneous elements of degrees  $x, y, z \in X$ , respectively.

The left and right unit constraints  $\lambda$  and  $\rho$ , respectively, are defined by

$$\lambda_V := \mathbb{K} \otimes V \xrightarrow{\sim} V : 1 \otimes v \mapsto \omega(e, e, x)^{-1}v,$$

and

$$\rho_V := V \otimes \mathbb{K} \xrightarrow{\sim} V : v \otimes 1 \mapsto \omega(x, e, e)v.$$

where  $V \in \text{Vec}_G^\omega$  and  $v \in V$  is a homogeneous element of degree  $x \in X$ .

Every pointed category is equivalent to some  $\text{Vec}_X^\omega$ .

**2.7. Crossed Modules.** Recall that a (*finite*) *crossed module* is a triple  $(G, X, \partial)$ , where  $G$  and  $X$  are (finite) groups with  $G$  acting on  $X$  as automorphisms, denoted  $(g, x) \mapsto {}^g x$ , and  $\partial : X \rightarrow G$  is a homomorphism satisfying

$$\partial(x) x' = x x' x^{-1}, \quad x, x' \in X,$$

and

$$\partial({}^g x) = g \partial(x) g^{-1}, \quad g \in G, x \in X.$$

Note that  $\text{Ker } \partial$  is a central subgroup of  $X$ .

A homomorphism of crossed modules  $(G, X, \partial) \rightarrow (G', X', \partial')$  is a pair of group homomorphisms  $(f : G \rightarrow G', F : X \rightarrow X')$  such that  $\partial' \circ F = f \circ \partial$  and  $F({}^g x) = {}^{f(g)} F(x)$ ,  $g \in G$ . We say that  $(f, F)$  is an *isomorphism* if both  $f$  and  $F$  are isomorphisms.

### 3. QUASI-ABELIAN THIRD COHOMOLOGY OF CROSSED MODULES

Let  $A$  be an abelian group. Eilenberg and Mac Lane [EM1, EM2, ML] argue that the cohomology groups  $H^n(A, \mathbb{K}^\times)$  are inappropriate since then do not take into account the abelianess of  $A$ , so should be replaced by groups  $H_{ab}^n(A, \mathbb{K}^\times)$ . (For the cohomology theory for crossed modules, see [W].) Below we recall the definition of  $H_{ab}^3(A, \mathbb{K}^\times)$ .

An *abelian 3-cocycle* on  $A$  is a pair  $(\omega, c)$ , where  $\omega$  is a normalized 3-cocycle on  $A$ , i.e.,

$$\begin{aligned} \omega(x, y, z) &= 1, \text{ if } x, y, \text{ or } z \text{ is identity,} \\ \omega(x, y, z)\omega(w, xy, z)\omega(w, x, y) &= \omega(w, x, yz)\omega(wx, y, z), \end{aligned}$$

for all  $w, x, y, z \in A$ , and  $c$  is a 2-cochain on  $A$  (i.e.,  $c \in C^2(A, \mathbb{K}^\times)$ ) satisfying the following equations:

$$\begin{aligned} c(xy, z) &= \frac{\omega(x, y, z)\omega(z, x, y)}{\omega(x, z, y)} c(x, z) c(y, z), \\ c(x, yz) &= \frac{\omega(y, x, z)}{\omega(x, y, z)\omega(y, z, x)} c(x, y) c(x, z) \end{aligned}$$

for all  $x, y, z \in A$ .

Abelian 3-cocycles on  $A$  form an abelian group, denoted by  $Z_{ab}^3(A, \mathbb{K}^\times)$ , under pointwise multiplication. The group of coboundaries is defined by

$$B_{ab}^3(A, \mathbb{K}^\times) := \left\{ \left( d\eta, (x, y) \mapsto \frac{\eta(y, x)}{\eta(x, y)} \right) \mid \text{normalized } \eta \in C^2(G, \mathbb{K}^\times) \right\},$$

which is a subgroup of  $Z_{ab}^3(A, \mathbb{K}^\times)$ . The quotient  $Z_{ab}^3(A, \mathbb{K}^\times)/B_{ab}^3(A, \mathbb{K}^\times)$  is the *abelian third cohomology* of  $A$  denoted  $H_{ab}^3(A, \mathbb{K}^\times)$ .

**Remark 3.1.** The group  $H_{ab}^3(A, \mathbb{K}^\times)$  is isomorphic to the group of quadratic forms on  $A$ , see [ML].

**Definition 3.2.** An abelian 3-cocycle  $(\omega, c)$  on  $A$  is *nondegenerate* if the symmetric bicharacter

$$A \times A \rightarrow \mathbb{K}^\times : (x, y) \mapsto c(y, x)c(x, y)$$

is nondegenerate.

In [O], C. Ospel generalized the notion of abelian third cohomology in the following way. Let  $G$  be a (not necessarily abelian) group. A *quasi-abelian 3-cocycle* on  $G$  is a pair  $(\omega, c)$ , where  $\omega$  is a 3-cocycle on  $G$  and  $c$  is a 2-cochain on  $G$  (i.e.,  $c \in C^2(G, \mathbb{K}^\times)$ ) satisfying the following equations:

$$\begin{aligned} \omega(gxg^{-1}, gyg^{-1}, gzg^{-1}) &= \omega(x, y, z), \\ c(gxg^{-1}, gyg^{-1}) &= c(x, y), \\ c(xy, z) &= \frac{\omega(x, y, z)\omega(xyz(xy)^{-1}, x, y)}{\omega(x, yzy^{-1}, y)}c(x, yzy^{-1})c(y, z), \\ c(x, yz) &= \frac{\omega(xyx^{-1}, x, z)}{\omega(x, y, z)\omega(y, z, x)}c(x, y)c(x, z), \end{aligned}$$

for all  $g, x, y, z \in G$ .

**Note 3.3.** The third equation above appeared in a slightly different but equivalent form in [O].

Quasi-abelian 3-cocycles on  $G$  form an abelian group, denoted by  $Z_{qa}^3(G, \mathbb{K}^\times)$ , under pointwise multiplication. The group of coboundaries is defined by

$$B_{qa}^3(G, \mathbb{K}^\times) := \left\{ \left( d(\eta), (x, y) \mapsto \frac{\eta(y, x)}{\eta(x, y)} \right) \mid \text{conjugation-invariant } \eta \in C^2(G, \mathbb{K}^\times) \right\},$$

which is a subgroup of  $Z_{qa}^3(G, \mathbb{K}^\times)$ . The quotient  $Z_{qa}^3(G, \mathbb{K}^\times)/B_{qa}^3(G, \mathbb{K}^\times)$  is the *quasi-abelian third cohomology* of  $G$  denoted  $H_{qa}^3(G, \mathbb{K}^\times)$ . When  $G$  is abelian, quasi-abelian cohomology reduces to abelian cohomology:  $H_{ab}^3(G, \mathbb{K}^\times) = H_{qa}^3(G, \mathbb{K}^\times)$ .

We extend Ospel's quasi-abelian cohomology for groups to cover crossed modules, as follows. We allow  $G$  to act on an arbitrary group  $X$  (not just  $X = G$ ). The first condition  $\omega^g = \omega$  in Ospel's definition is replaced by the condition ' $\omega^g$  is cohomologous to  $\omega$  via  $\mu_g$ '. The second condition  $c^g = c$  is extended similarly, as are the other conditions. This results in the following definition, whose main motivation is the classification of crossed pointed categories (see Section 4).

**Definition 3.4.** A *quasi-abelian 3-cocycle* on a crossed module  $\mathcal{X} = (G, X, \partial)$  is a quadruple  $(\omega, \gamma, \mu, c)$  where

$$(a) \quad \omega \in Z^3(X, \mathbb{K}^\times),$$

$$(b) \quad \gamma \in Z^2(G, C^1(X, \mathbb{K}^\times)),$$

$\mu \in C^1(G, C^2(X, \mathbb{K}^\times))$  satisfying

$$(c) \quad d(\mu_g) = \frac{\omega^g}{\omega}, \quad g \in G,$$

that is,

$$\frac{\mu_g(y, z)\mu_g(x, yz)}{\mu_g(xy, z)\mu_g(x, y)} = \frac{\omega^g(x, y, z)}{\omega(x, y, z)}, \quad g \in G, x, y, z \in X,$$

$$(d) \quad d(\gamma_{g,h}) = (d\mu)_{g,h}, \quad g, h \in G,$$

that is,

$$\frac{\gamma_{g,h}(x)\gamma_{g,h}(y)}{\gamma_{g,h}(xy)} = \frac{\mu_g({}^h x, {}^h y)\mu_h(x, y)}{\mu_{gh}(x, y)}, \quad g, h \in G, x, y \in X$$

and  $c \in C^2(X, \mathbb{K}^\times)$  satisfying

$$(e) \quad \frac{c^g(x, y)}{c(x, y)} = \frac{\mu_g(xy x^{-1}, x)}{\mu_g(x, y)} \frac{\gamma_{g\partial(x)g^{-1}, g}(y)}{\gamma_{g, \partial(x)}(y)}, \quad g \in G, x, y \in X,$$

$$(f) \quad c(xy, z) = \frac{\omega(x, y, z)\omega((xy)z(xy)^{-1}, x, y)}{\omega(x, yzy^{-1}, y)\gamma_{\partial(x), \partial(y)}(z)} c(x, yzy^{-1})c(y, z), \quad x, y, z \in X,$$

and

$$(g) \quad c(x, yz) = \frac{\omega(xy x^{-1}, x, z)}{\omega(x, y, z)\omega(xy x^{-1}, xzx^{-1}, x)\mu_{\partial(x)}(y, z)} c(x, y)c(x, z), \quad x, y, z \in X.$$

**Note 3.5.** Some remarks about the notation:  $C^n$  denotes the space of  $n$ -cochains,  $Z^n$  denotes the space of  $n$ -cocycles, and  $d$  is the usual *differential operator* [B]. (We note that the definition of  $d$  depends on whether the module under consideration is left or right.) The action of  $G$  on  $X$ ,  $(g, x) \mapsto {}^g x$ , induces a right action of  $G$  on  $C^n(X, \mathbb{K}^\times)$  by translations. The map  $c^g \in C^2(X, \mathbb{K}^\times)$  is defined by  $c^g(x, y) := c({}^g x, {}^g y)$  and the map  $\omega^g$  is defined similarly.

Quasi-abelian 3-cocycles on a crossed module  $\mathcal{X} = (G, X, \partial)$  form an abelian group, denoted by  $Z_{qa}^3(\mathcal{X}, \mathbb{K}^\times)$ , under pointwise multiplication.

We define the group of coboundaries by

$$B_{qa}^3(\mathcal{X}, \mathbb{K}^\times) := \left\{ \left( d\eta, d\beta, g \mapsto d(\beta_g) \frac{\eta^g}{\eta}, (x, y) \mapsto \beta_{\partial(x)}(y) \frac{\eta(xy x^{-1}, x)}{\eta(x, y)} \right) \middle| \begin{array}{l} \eta \in C^2(X, \mathbb{K}^\times), \\ \beta \in C^1(G, C^1(X, \mathbb{K}^\times)) \end{array} \right\}.$$

A direct computation shows that  $B_{qa}^3(\mathcal{X}, \mathbb{K}^\times) \subseteq Z_{qa}^3(\mathcal{X}, \mathbb{K}^\times)$ .

**Definition 3.6.** The *quasi-abelian third cohomology* of a crossed module  $\mathcal{X}$ , denoted  $H_{qa}^3(\mathcal{X}, \mathbb{K}^\times)$ , is the quotient of  $Z_{qa}^3(\mathcal{X}, \mathbb{K}^\times)$  by  $B_{qa}^3(\mathcal{X}, \mathbb{K}^\times)$ .

**Remark 3.7.** Let  $G$  be a group. Consider the crossed module  $\mathcal{G} = (G, G, \text{id}_G)$ , where  $G$  acts on itself by conjugation.



(i) There is a homomorphism  $H_{qa}^3(G, \mathbb{K}^\times) \rightarrow H_{qa}^3(\mathcal{G}, \mathbb{K}^\times)$  induced from

$$Z_{qa}^3(G, \mathbb{K}^\times) \rightarrow Z_{qa}^3(\mathcal{G}, \mathbb{K}^\times) : (\omega, c) \mapsto (\omega, 1, 1, c).$$

(ii) There also exists a homomorphism  $H^3(G, \mathbb{K}^\times) \rightarrow H_{qa}^3(\mathcal{G}, \mathbb{K}^\times)$  (see Lemma 6.3).

**Definition 3.8.** A quasi-abelian 3-cocycle  $(\omega, \gamma, \mu, c)$  is normalized if

$$\begin{aligned} \omega(x, y, z) &= 1, \text{ if } x, y, \text{ or } z \text{ is identity,} & \gamma_{g,h}(x) &= 1, \text{ if } g, h, \text{ or } x \text{ is identity,} \\ \mu_g(x, y) &= 1, \text{ if } x, y, \text{ or } g \text{ is identity,} & c(x, y) &= 1, \text{ if } x \text{ or } y \text{ is identity.} \end{aligned}$$

**Note 3.9.** Every quasi-abelian 3-cocycle is cohomologous to a normalized one.

Let  $(\omega, \gamma, \mu, c)$  be a normalized quasi-abelian 3-cocycle on a crossed module  $(G, X, \partial)$ . Then  $(\omega|_{\text{Ker } \partial}, c|_{\text{Ker } \partial})$  is an abelian 3-cocycle on the (abelian) group  $\text{Ker } \partial$ .

**Definition 3.10.** A normalized quasi-abelian 3-cocycle  $(\omega, \gamma, \mu, c)$  on a crossed module  $(G, X, \partial)$  is *nondegenerate* if the abelian 3-cocycle  $(\omega|_{\text{Ker } \partial}, c|_{\text{Ker } \partial})$  on the (abelian) group  $\text{Ker } \partial$  is nondegenerate.

Any homomorphism  $(f, F) : (G', X', \partial') = \mathcal{X}' \rightarrow \mathcal{X} = (G, X, \partial)$  of crossed modules induces a homomorphism

$$Z_{qa}^3(\mathcal{X}, \mathbb{K}^\times) \rightarrow Z_{qa}^3(\mathcal{X}', \mathbb{K}^\times) : (\omega, \gamma, \mu, c) \mapsto (\omega, \gamma, \mu, c)^{(f, F)}$$

where

$$(\omega, \gamma, \mu, c)^{(f, F)} = (\omega \circ F^{\times 3}, (g, h) \mapsto \gamma_{f(g), f(h)} \circ F, g \mapsto \mu_{f(g)} \circ F^{\times 2}, c \circ F^{\times 2}).$$

It is straight-forward to verify that the above homomorphism preserves coboundaries, thereby it provides a homomorphism  $H_{qa}^3(\mathcal{X}, \mathbb{K}^\times) \rightarrow H_{qa}^3(\mathcal{X}', \mathbb{K}^\times)$ . Consequently, for any crossed module  $\mathcal{X}$  there is a natural action of the group of automorphisms  $\text{Aut}(\mathcal{X})$  of  $\mathcal{X}$  on  $H_{qa}^3(\mathcal{X}, \mathbb{K}^\times)$ .

#### 4. CLASSIFICATION OF CROSSED POINTED CATEGORIES

In this section we classify crossed pointed categories in terms of quasi-abelian third cohomology of crossed modules.

**4.1. Construction of a crossed pointed category from a quasi-abelian 3-cocyle on a crossed module.** Given a quasi-abelian 3-cocycle  $(\omega, \gamma, \mu, c)$  on a finite crossed module  $(G, X, \partial)$ , we associate to it a crossed pointed category  $(\mathcal{C}, G, T, \tilde{\gamma}, \iota, \tilde{\mu}, \nu, \tilde{c})$  as follows.

As a fusion category  $\mathcal{C} = \text{Vec}_X^\omega$ . For each  $g \in G$ , let  $\mathcal{C}_g$  denote the full abelian subcategory consisting of objects of  $\text{Vec}_X^\omega$  supported on  $\partial^{-1}(g) \subset X$ , i.e., objects of  $\mathcal{C}_g$  are defined to be finite-dimensional  $\partial^{-1}(g)$ -graded vector spaces (we set  $\mathcal{C}_g := \{0\}$  if  $\partial^{-1}(g)$  is empty). This defines a  $G$ -grading of  $\mathcal{C}$ :  $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ .

Next we define a functor  $T : \underline{G} \rightarrow \text{Aut}_\otimes(\mathcal{C}) : g \mapsto T_g$  as follows. Let  $V \in \text{Vec}_X^\omega$  be a homogeneous object of degree  $x \in X$ . The functor  $T_g : \text{Vec}_X^\omega \xrightarrow{\sim} \text{Vec}_X^\omega$  is defined by  $T_g(V) := V$  (as a vector space) and the degree of  $T_g(V)$  is defined to be  ${}^g x$ . The  $T_g$ 's are extended to nonhomogeneous objects and morphisms in the obvious way.

The tensor structure  $\tilde{\gamma}$  on the functor  $T : \underline{G} \rightarrow \text{Aut}_\otimes(\mathcal{C})$  is defined by

$$\gamma_{g,h}(x) \text{id}_V =: \tilde{\gamma}_{g,h}(V) : T_{gh}(V) \xrightarrow{\sim} (T_g \circ T_h)(V),$$

for all homogeneous objects  $V \in \text{Vec}_X^\omega$  of degree  $x \in X$ , and  $g, h \in G$ .

The unit-preserving structure  $\iota : T_e \xrightarrow{\sim} \text{id}_{\mathcal{C}}$  on the functor  $T : \underline{G} \rightarrow \text{Aut}_{\otimes}(\mathcal{C})$  is defined by

$$\gamma_{e,e}^{-1}(x) \text{id}_V =: \iota(V) : T_e(V) \xrightarrow{\sim} \text{id}_{\mathcal{C}}(V),$$

for all homogeneous objects  $V \in \text{Vec}_X^\omega$  of degree  $x \in X$ .

The tensor structure  $\tilde{\mu}_g$  on the functor  $T_g : \text{Vec}_X^\omega \xrightarrow{\sim} \text{Vec}_X^\omega$ ,  $g \in G$ , is defined by

$$\mu_g(x, y) \text{id}_{V \otimes_{\mathbb{K}} W} =: \tilde{\mu}_g(V, W) : T_g(V \otimes W) \xrightarrow{\sim} T_g(V) \otimes T_g(W),$$

for all homogeneous objects  $V, W \in \text{Vec}_X^\omega$  of degrees  $x, y \in X$ , respectively.

The unit-preserving structure  $\nu_g$  on the functor  $T_g : \text{Vec}_X^\omega \xrightarrow{\sim} \text{Vec}_X^\omega$ ,  $g \in G$ , is defined by

$$\mu_g^{-1}(e, e) \text{id}_{\mathbb{K}} =: \nu_g : T_g(\mathbb{K}) \xrightarrow{\sim} \mathbb{K}.$$

For  $V, W \in \text{Vec}$ , let  $\tau_{V,W}$  denote the flip operator  $V \otimes_{\mathbb{K}} W \xrightarrow{\sim} W \otimes_{\mathbb{K}} V : v \otimes_{\mathbb{K}} w \mapsto w \otimes_{\mathbb{K}} v$ . The  $G$ -braiding  $\tilde{c}$  is defined by

$$c(x, y) \tau_{V,W} =: \tilde{c}(V, W) : V \otimes W \xrightarrow{\sim} T_g(W) \otimes V,$$

for all homogeneous objects  $V, W \in \text{Vec}_X^\omega$  of degrees  $x, y \in X$ . (Here  $g = \partial(x)$ .)

The crossed module axioms of  $(G, X, \partial)$  and the quasi-abelian 3-cocycle axioms of  $(\omega, \gamma, \mu, c)$  together ensure that the necessary axioms of a crossed category are satisfied. Specifically, Condition (c) of Definition 3.4 ensures that  $\tilde{\mu}_g$  is a tensor structure on the functor  $T_g$  defined above. Condition (d) of Definition 3.4 ensures that  $\tilde{\gamma}$  is a tensor structure on the functor  $T$ . Conditions (e)-(g) of Definition 3.4 correspond to Axioms (i)-(iii) of Definition 3.4, respectively.

We will denote the crossed pointed category constructed above by

$$\mathcal{C}(\omega, \gamma, \mu, c).$$

**Remark 4.1.** The trivial component  $\mathcal{C}(\omega, \gamma, \mu, c)_e$  of  $\mathcal{C}(\omega, \gamma, \mu, c)$  (under the  $G$ -grading) is a braided fusion category. As a fusion category,  $\mathcal{C}(\omega, \gamma, \mu, c)_e = \text{Vec}_{\text{Ker } \partial}^{\omega|_{\text{Ker } \partial}}$ . Suppose that the quasi-abelian 3-cocycle  $(\omega, \gamma, \mu, c)$  is normalized. Then the braiding on the trivial component is given by

$$V \otimes W \rightarrow W \otimes V : v \otimes w \mapsto c(x, y) w \otimes v,$$

for all homogeneous objects  $V, W \in \text{Vec}_{\text{Ker } \partial}^{\omega|_{\text{Ker } \partial}}$  of degrees  $x, y \in \text{Ker } \partial$ . Clearly, the braided fusion category  $\mathcal{C}(\omega, \gamma, \mu, c)_e$  is nondegenerate if and only if the quasi-abelian 3-cocycle  $(\omega, \gamma, \mu, c)$  is nondegenerate in the sense of Definition 3.10.

## 4.2. Classification.

**Proposition 4.2.** *Let  $\mathcal{C}(\omega, \gamma, \mu, c)$  and  $\mathcal{C}(\omega', \gamma', \mu', c')$  be crossed pointed categories constructed in the preceding subsection. Then  $\mathcal{C}(\omega, \gamma, \mu, c) \cong \mathcal{C}(\omega', \gamma', \mu', c')$  as crossed categories if and only if there exists an isomorphism  $(f, F)$  of the underlying (finite) crossed modules such that the quasi-abelian 3-cocycles  $(\omega', \gamma', \mu', c')^{(f, F)}$  and  $(\omega, \gamma, \mu, c)$  are cohomologous.*

*Proof.* Suppose  $(G, X, \partial)$  and  $(G', X', \partial')$  are the underlying (finite) crossed modules of  $\mathcal{C}(\omega, \gamma, \mu, c)$  and  $\mathcal{C}(\omega', \gamma', \mu', c')$ , respectively. Let  $(f, F)$  be an isomorphism from  $(G, X, \partial)$  to  $(G', X', \partial')$  such that the quasi-abelian 3-cocycles  $(\omega', \gamma', \mu', c')^{(f, F)}$  and  $(\omega, \gamma, \mu, c)$  are cohomologous via  $(\eta, \beta)$  (see Section 3). In what follows we will construct an equivalence  $(f, \tilde{F}, \tilde{\eta}, \eta_0, \tilde{\beta})$  of crossed categories from  $\mathcal{C}(\omega, \gamma, \mu, c)$  to  $\mathcal{C}(\omega', \gamma', \mu', c')$  (see Definition 2.6).

Recall that as fusion categories,  $\mathcal{C}(\omega, \gamma, \mu, c) = \text{Vec}_X^\omega$  and  $\mathcal{C}(\omega', \gamma', \mu', c') = \text{Vec}_{X'}^{\omega'}$ . Let  $V \in \text{Vec}_X^\omega$  be a homogeneous object of degree  $x \in X$ . Define a functor  $\tilde{F} : \text{Vec}_X^\omega \rightarrow \text{Vec}_{X'}^{\omega'}$ , by  $\tilde{F}(V) := V$  (as a vector space) and the degree of  $\tilde{F}(V)$  is defined to be  $F(x)$ . The functor  $\tilde{F}$  extends to nonhomogeneous objects and morphisms in the obvious way.

The tensor structure  $\tilde{\eta}$  on the functor  $\tilde{F}$  is defined by

$$\eta(x, y) \text{id}_{V \otimes W} =: \tilde{\eta}(V, W) : \tilde{F}(V \otimes W) \xrightarrow{\sim} \tilde{F}(V) \otimes \tilde{F}(W),$$

for all homogeneous objects  $V, W \in \text{Vec}_X^\omega$  of degrees  $x, y \in X$ , respectively.

The definition of the unit-preserving structure  $\eta_0$  on  $\tilde{F}$  is obvious. It is easy to verify that  $(\tilde{F}, \tilde{\eta}, \eta_0)$  is an equivalence of tensor categories.

Next we define isomorphisms  $\tilde{\beta}_g : F \circ T_g \xrightarrow{\sim} T'_{f(g)} \circ F, g \in G$ , of tensor functors by

$$\beta_g(x) \text{id}_V =: \tilde{\beta}_g(V) : (\tilde{F} \circ T_g)(V) \xrightarrow{\sim} (T'_{f(g)} \circ \tilde{F})(V),$$

for all homogeneous objects  $V \in \text{Vec}_X^\omega$  of degree  $x \in X$ .

It is easy to verify that axioms (i)-(iii) of Definition 2.6 are satisfied. This shows that  $\mathcal{C}(\omega, \gamma, \mu, c) \cong \mathcal{C}(\omega', \gamma', \mu', c')$ , as crossed categories.

The converse is clear from the above construction.  $\square$

**Remark 4.3.** The above proposition, in particular, shows that if the quasi-abelian 3-cocycles  $(\omega, \gamma, \mu, c)$  and  $(\omega', \gamma', \mu', c')$  (on the same crossed module  $(G, X, \partial)$ ) are cohomologous, then the corresponding crossed pointed categories  $\mathcal{C}(\omega, \gamma, \mu, c)$  and  $\mathcal{C}(\omega', \gamma', \mu', c')$  are equivalent.

Recall that for any crossed module  $\mathcal{X}$  there is a natural action of  $\text{Aut}(\mathcal{X})$  on the quasi-abelian third cohomology  $H_{qa}^3(\mathcal{X}, \mathbb{K}^\times)$  of  $\mathcal{X}$  (see Section 3).

**Theorem 4.4.** *Crossed pointed categories are classified, up to equivalence, by orbits of the quasi-abelian third cohomology  $H_{qa}^3(\mathcal{X}, \mathbb{K}^\times)$  (of a finite crossed module  $\mathcal{X}$ ) under the action of  $\text{Aut}(\mathcal{X})$ .*

*Proof.* Every crossed pointed category is equivalent to some  $\mathcal{C}(\omega, \gamma, \mu, c)$  with underlying (finite) crossed module  $\mathcal{X}$ . The theorem now follows from Proposition 4.2.  $\square$

## 5. EQUIVARIANTIZATION OF $\mathcal{C}(\omega, \gamma, \mu, c)$

Throughout this section, let  $(\omega, \gamma, \mu, c)$  be a normalized quasi-abelian 3-cocycle on a finite crossed module  $(G, X, \partial)$ . In Subsection 4.1 we associated to  $(\omega, \gamma, \mu, c)$  a crossed pointed category  $\mathcal{C}(\omega, \gamma, \mu, c)$ . Our goal in this section is to apply the *equivariantization* process to  $\mathcal{C}(\omega, \gamma, \mu, c)$  and study the resulting braided fusion category.

**5.1. Description.** Recall that as a fusion category,  $\mathcal{C}(\omega, \gamma, \mu, c) = \text{Vec}_X^\omega$ . We begin with the following.

**Proposition 5.1.** *An object of the equivariantization category  $\mathcal{C}(\omega, \gamma, \mu, c)^G$  is a  $X$ -graded vector space  $V$  together with a twisted action  $\triangleright$  of  $G$  on  $V$  which is compatible with the grading, in the sense of*

$$(5) \quad \begin{aligned} gh \triangleright v &= \gamma_{g,h}(x)(g \triangleright (h \triangleright v)) \\ e \triangleright v &= v, \quad \text{degree}(g \triangleright v) = {}^g x, \end{aligned}$$

for all  $v \in V$  homogeneous of degree  $x \in X$  and  $g, h \in G$ . Morphisms in the category are linear maps preserving the twisted action and grading. The twisted action of  $G$  on the tensor product is given by

$$(6) \quad g \triangleright (v \otimes w) = \mu_g(x, y)(g \triangleright v \otimes g \triangleright w),$$

for homogeneous  $v, w$  of degrees  $x, y \in X$ , respectively. The associativity constraint on the category is given by

$$(u \otimes v) \otimes w \mapsto \omega(x, y, z)u \otimes (v \otimes w),$$

for all homogeneous  $u, v, w$  of degrees  $x, y, z \in X$ . The braiding on the category is given by

$$(7) \quad v \otimes w \mapsto c(x, y)(\partial(x) \triangleright w \otimes v),$$

for all homogeneous  $v, w$  of degrees  $x, y \in X$ .

*Proof.* The action  $\triangleright$  referred to in the statement of the proposition corresponds to equivariant structure (1). Equation 5 corresponds to (2). The definition of the tensor product (6) comes from (3) and the definition of the braiding (7) comes (4).  $\square$

**Remark 5.2.** There is a simple special case of the above description. Namely, take the quasi-abelian 3-cocycle  $(\omega, \gamma, \mu, c)$  (on the finite crossed module  $(G, X, \partial)$ ) to be trivial. Then the corresponding equivariantization category  $\mathcal{C}(1, 1, 1, 1)^G$  admits a simple description: objects of this category are  $G$ -equivariant vector bundles on  $X$ . We note that this braided fusion category was considered in [Ba]. This category is not nondegenerate in general: by Proposition 5.6 it is nondegenerate if and only if  $\partial$  is an isomorphism. In this case, the category is equivalent to  $D(G)\text{-Mod}$ , as a braided fusion category.

**Theorem 5.3.** *Every braided group-theoretical category is equivalent to  $\mathcal{C}(\xi)^G$ , for some normalized quasi-abelian 3-cocycle  $\xi$  on a finite crossed module  $(G, X, \partial)$ .*

*Proof.* This follows from [NNW], where it was shown that every braided group-theoretical category is the equivariantization of a pointed category.  $\square$

**Lemma 5.4.** *For any  $x \in X$ , let  $\text{Stab}_G(x)$  denote the stabilizer of  $x$  in  $G$ , i.e.,  $\text{Stab}_G(x) = \{g \in G \mid {}^g x = x\}$ . Define  $\phi_x : \text{Stab}_G(x) \times \text{Stab}_G(x) \rightarrow \mathbb{K}^\times$  by*

$$\phi_x(g, h) := \gamma_{g,h}(x), \quad g, h \in \text{Stab}_G(x).$$

*Then  $\phi_x$  is a 2-cocycle on  $\text{Stab}_G(x)$ .*

*Proof.* The Condition (b) on  $\gamma$  in Definition 3.4 means that

$$\gamma_{h,k}(x)\gamma_{g,hk}(x) = \gamma_{gh,k}(x)\gamma_{g,h}(x),$$

for all  $g, h, k \in G, x \in X$ . Restricting to  $\text{Stab}_G(x)$  we get the stated assertion.  $\square$

Let  $R$  denote a complete set of representatives of orbits of  $X$  under the action of  $G$ .

**Proposition 5.5.** *The set of isomorphism classes of simple objects of  $\mathcal{C}(\omega, \gamma, \mu, c)^G$  is in bijection with isomorphism classes of the set*

$$(8) \quad \Gamma := \{(a, V) \mid a \in R, V \text{ is an irreducible module over } \mathbb{K}_{\phi_a}[\text{Stab}_G(a)]\},$$

where  $\phi_a$  is the 2-cocycle defined in Lemma 5.4.

*Proof.* Let  $\text{Irr}(\mathcal{C}(\omega, \gamma, \mu, c)^G)$  denote the set of simple objects of  $\mathcal{C}^G$ . We will define a map

$$(9) \quad \Gamma \rightarrow \text{Irr}(\mathcal{C}(\omega, \gamma, \mu, c)^G)$$

and show that it induces a bijection between the isomorphism classes of the source and target sets. Let  $g_1, g_2, \dots$  be coset representatives of  $\text{Stab}_G(a)$  in  $G$ . Pick any  $(a, V) \in \Gamma$ . We define the map (9) by

$$(10) \quad (a, V) \mapsto \tilde{V} = \oplus_{g_i} V_{g_i a},$$

where  $V_{g_i a} = V$  as a vector space and  $\text{degree}(V_{g_i a}) = g_i a$ . The twisted action of  $G$  on  $\tilde{V}$  is given by

$$(11) \quad h \triangleright v := \frac{\gamma_{g_j, t}(a)}{\gamma_{h, g_i}(a)} (t \triangleright v),$$

for all  $v \in \tilde{V}$  homogeneous of degree  $g_i a$  with  $t \in \text{Stab}_G(a)$  uniquely determined by the equation  $hg_i = g_j t$ . The degree of  $h \triangleright v$  is defined to be  $g_j a$ .

To prove that the map (9) (defined via (10) and (11)) is well-defined we need to show that the action defined in (11) satisfies (5). This amounts to verifying that the scalar

$$\frac{\gamma_{g_k, st}(a)\gamma_{s, t}(a)}{\gamma_{gh, g_i}(a)}$$

is equal to the scalar

$$\frac{\gamma_{g, h}(g_i a)\gamma_{g_j, t}(a)\gamma_{g_k, s}(a)}{\gamma_{h, g_i}(a)\gamma_{g, g_j}(a)}$$

for all  $g, h \in G, s, t \in \text{Stab}_G(a)$  with  $hg_i = g_j t$  and  $gg_j = g_k s$ . The equality of the two scalars follows from applying Condition (b) on  $\gamma$  in Definition 3.4 successively to the quadruples  $(g, h, g_i, a), (g, g_j, t, a), (g_k, s, t, a)$ .

We now show that the map (9) induces a bijection between isomorphism classes of the source and target sets. It is clear that the map (9) preserves isomorphic objects. Furthermore, the object in  $\text{Irr}(\mathcal{C}(\omega, \gamma, \mu, c)^G)$  corresponding to  $(a, V) \in \Gamma$  has FP-dimension

equal to  $|^Ga| \dim_{\mathbb{K}} V$ , where  $^Ga$  denotes the orbit containing  $a$ . The sum of squares of FP-dimensions of isomorphism classes of objects in the image of (9) is

$$\begin{aligned}
 \sum_{a \in R} \sum_{V \in \text{Irr}(\mathbb{K}_{\phi_a}[\text{Stab}_G(a)])} |^Ga|^2 (\dim V)^2 &= \sum_{a \in R} |^Ga|^2 |\text{Stab}_G(a)| \\
 (12) \qquad \qquad \qquad &= \sum_{a \in R} |^Ga| |G| \\
 &= |G| |X| \\
 &= \text{FPdim}(\mathcal{C}(\omega, \gamma, \mu, c)^G),
 \end{aligned}$$

completing the proof.  $\square$

**5.2. Twist and  $S$ -matrix.** As before,  $R$  denotes a complete set of representatives of orbits of  $X$  under the action of  $G$ . By Proposition 5.5, the simple objects of  $\mathcal{C}(\omega, \gamma, \mu, c)^G$  correspond to pairs  $(a, \chi)$ , where  $a \in R$  and  $\chi$  is an irreducible  $\phi_a$ -character of  $\text{Stab}_G(a)$ . Note that  $\mathcal{C}(\omega, \gamma, \mu, c)^G$  admits a canonical twist  $\theta$  with respect to which categorical dimensions coincide with FP-dimensions. The values of  $\theta$  on simple objects are given by

$$\theta_{(a, \chi)} = c(a, a) \frac{\chi(\partial(a))}{\deg \chi}.$$

A direct calculation shows that the  $S$ -matrix  $S$  is given by

$$S_{(a, \chi), (b, \chi')} = \sum_{\substack{x \in ({}^Ga) \\ y \in ({}^Gb) \cap C_X(x)}} c(x, y) c(y, x) \frac{\gamma_{g, \partial(g^{-1}y)}(a) \gamma_{h, \partial(h^{-1}x)}(b)}{\gamma_{\partial(y), g}(a) \gamma_{\partial(x), h}(b)} \chi(g^{-1}y) \chi'(h^{-1}x),$$

where in each summand  $g$  and  $h$  are defined by  ${}^ga = x$  and  ${}^hb = y$ . (Note that the choice of  $g$  and  $h$  does not affect the sum.)

**5.3. Modularity.** As before,  $(\omega, \gamma, \mu, c)$  is a normalized quasi-abelian 3-cocycle on a finite crossed module  $(G, X, \partial)$ . We have the following.

**Proposition 5.6.** *The braided fusion category  $\mathcal{C}(\omega, \gamma, \mu, c)^G$  is nondegenerate if and only if the homomorphism  $\partial$  is surjective and  $(\omega, \gamma, \mu, c)$  is nondegenerate in the sense of Definition 3.10.*

*Proof.* This follows immediately by combining Remark 2.5 and Remark 4.1.  $\square$

Assume that  $\partial$  is surjective and  $(\omega, \gamma, \mu, c)$  is nondegenerate. Then  $\mathcal{C}(\omega, \gamma, \mu, c)^G$  together with the canonical twist given in Subsection 5.2 is a modular category, i.e., the  $S$ -matrix described in Subsection 5.2 is invertible. In this situation, using the orthogonality relations for projective characters we obtain that the Gauss sum and central charge, respectively, of  $\mathcal{C}(\omega, \gamma, \mu, c)^G$  are given by

$$\begin{aligned}
 \tau(\mathcal{C}(\omega, \gamma, \mu, c)^G) &= |G| \sum_{a \in \text{Ker } \partial} c(a, a), \\
 \zeta(\mathcal{C}(\omega, \gamma, \mu, c)^G) &= \frac{1}{\sqrt{|\text{Ker } \partial|}} \sum_{a \in \text{Ker } \partial} c(a, a).
 \end{aligned}$$

**Note 5.7.** The sum  $\sum_{a \in \text{Ker } \partial} c(a, a)$  is the classical Gauss sum for the quadratic form  $a \mapsto c(a, a)$  on the abelian group  $\text{Ker } \partial$ .

**Remark 5.8.** Note that the category  $\mathcal{C}(\omega, \gamma, \mu, c)^G$  may admit other twists (besides the canonical one). In view of Theorem 5.3 and Proposition 5.6, a description of all twists on  $\mathcal{C}(\omega, \gamma, \mu, c)^G$  will imply a description of *all* modular group-theoretical categories. The former is easily obtained using Proposition 2.1.

## 6. QUASI-TRIANGULAR QUASI-HOPF ALGEBRA ARISING FROM QUASI-ABELIAN 3-COCYCLES ON CROSSED MODULES

Let  $(\omega, \gamma, \mu, c)$  be a normalized quasi-abelian 3-cocycle on a finite crossed module  $(G, X, \partial)$ . In the previous section we described the braided fusion category  $\mathcal{C}(\omega, \gamma, \mu, c)^G$ . This category is integral (i.e., the FP-dimensions of objects are integers), so there exists a finite-dimensional quasi-triangular quasi-Hopf algebra  $H$  such that  $\mathcal{C}(\omega, \gamma, \mu, c)^G \cong H\text{-Mod}$ , as braided fusion categories (see [ENO, Theorem 8.33] and [K, Section XV.2]). Our goal in this section is to describe such an  $H$ .

In what follows we associate to  $(\omega, \gamma, \mu, c)$  a finite-dimensional quasi-triangular quasi-Hopf algebra  $H(\omega, \gamma, \mu, c)$ , which may be viewed as a generalization of the twisted Drinfeld double of a finite group. Let  $H(\omega, \gamma, \mu, c)$  be a finite-dimensional vector space with a basis  $\{t_{xg}\}_{(x,g) \in X \times G}$  indexed by the set  $X \times G$ . Define a product on  $H(\omega, \gamma, \mu, c)$  by

$$(13) \quad (t_x g)(t_y h) := \delta_{x, h_y} \gamma_{g, h}(y)^{-1} t_y(gh).$$

This product admits a unit

$$(14) \quad 1 = \sum_{x \in X} t_x e.$$

Define a coproduct  $\Delta : H(\omega, \gamma, \mu, c) \rightarrow H(\omega, \gamma, \mu, c) \otimes H(\omega, \gamma, \mu, c)$  and counit  $\varepsilon : H(\omega, \gamma, \mu, c) \rightarrow \mathbb{K}$  by

$$(15) \quad \Delta(t_x g) := \sum_{a, b \in X : ab = x} \mu_g(a, b) t_a g \otimes t_b g$$

and

$$(16) \quad \varepsilon(t_x g) := \delta_{x, e}.$$

Also, set

$$(17) \quad \Phi := \sum_{x, y, z \in X} \omega(x, y, z) t_x e \otimes t_y e \otimes t_z e,$$

$$(18) \quad R := \sum_{x, y \in X} c(x, y) t_x e \otimes t_y \partial(x),$$

and

$$(19) \quad \alpha := 1, \quad \beta := \sum_{x \in X} \omega(x^{-1}, x, x^{-1}) t_x e.$$

Finally, define a linear map  $S : H(\omega, \gamma, \mu, c) \rightarrow H(\omega, \gamma, \mu, c)$  by

$$(20) \quad S(t_x g) := \frac{\gamma_{g^{-1}, g}(x^{-1})}{\mu_g(x, x^{-1})} t_{g x^{-1}} g^{-1}.$$

**Proposition 6.1.** *The product (13), unit (14), coproduct  $\Delta$  (15), counit  $\varepsilon$  (16), Drinfeld associator  $\Phi$  (17), and anti-automorphism  $S$  (20) make  $H(\omega, \gamma, \mu, c)$  a quasi-triangular quasi-Hopf algebra with universal  $R$ -matrix  $R$  (18) in the sense of [K, Definitions 1.1, 2.1, and 5.1].*

*Proof.* The proof is completely similar to the one for the twisted Drinfeld double of a finite group: Associativity of the product is equivalent to the equality

$$\gamma_{h,k}(x) \gamma_{g,hk}(x) = \gamma_{gh,k}(x) \gamma_{g,h}(^k x), \quad g, h, k \in G, x \in X,$$

which holds by Axiom (b) in Definition 3.4. Quasi-coassociativity of the coproduct is equivalent to the equality

$$\frac{\mu_g(y, z) \mu_g(x, yz)}{\mu_g(xy, z) \mu_g(x, y)} = \frac{\omega(^g x, ^g y, ^g z)}{\omega(x, y, z)}, \quad g \in G, x, y, z \in X,$$

which holds by Axiom (c) in Definition 3.4. That the coproduct is a morphism of algebras is equivalent to the equality

$$\frac{\gamma_{g,h}(x) \gamma_{g,h}(y)}{\gamma_{g,h}(xy)} = \frac{\mu_g(^h x, ^h y) \mu_h(x, y)}{\mu_{gh}(x, y)}, \quad g, h \in G, x, y \in X,$$

which holds by Axiom (d) in Definition 3.4.

We note that the inverse of the  $R$ -matrix  $R$  is

$$R^{-1} = \sum_{x, y \in X} c(x, x^{-1} y x)^{-1} \gamma_{\partial(x), \partial(x^{-1})}(y)^{-1} t_x e \otimes t_y \partial(x^{-1}).$$

The  $R$ -matrix axioms on  $R$  hold due to Axioms (e)-(g) in Definition 3.4.

Finally, Axioms (a)-(d) in Definition 3.4 ensure that  $S$  is indeed an anti-automorphism that satisfies the required axioms.  $\square$

**Proposition 6.2.** *Let  $(\omega, \gamma, \mu, c)$  be a normalized quasi-abelian 3-cocycle on a finite crossed module  $(G, X, \partial)$ . The categories  $\mathcal{C}(\omega, \gamma, \mu, c)^G$  (see Section 5) and  $H(\omega, \gamma, \mu, c)\text{-Mod}$  are equivalent as braided fusion categories.*

*Proof.* Let  $V$  be a (left) module over  $H(\omega, \gamma, \mu, c)$ , with action denoted by “ $\cdot$ ”. Note that  $V$  admits an  $X$ -grading:  $V = \bigoplus_{x \in X} V_x$ , where  $V_x = (t_x e) \cdot V$ . Define a twisted action of  $G$  on  $V$  by

$$g \triangleright v := (t_x g) \cdot v,$$

for all  $v \in V$  homogeneous of degree  $x \in X$ . Observe that the degree of  $g \triangleright v$  is  $^g x$ , as  $(t_x g)(t_x e) = (t_{gx} e)(t_x g)$ . The aforementioned action is twisted in the sense that

$$gh \triangleright v = \gamma_{g,h}(x)(g \triangleright (h \triangleright v)).$$

Note that the twisted action of  $G$  completely determines the action of  $H(\omega, \gamma, \mu, c)$  on the module  $V$ . The associativity constraint on the category  $H(\omega, \gamma, \mu, c)\text{-Mod}$  (which is defined using the Drinfeld associator  $\Phi$  (17)) is given by

$$(u \otimes v) \otimes w \mapsto \omega(x, y, z) u \otimes (v \otimes w),$$



for all homogeneous  $u, v, w$  of degrees  $x, y, z \in X$ . The braiding on the category  $H(\omega, \gamma, \mu, c)\text{-Mod}$  (which is defined using the  $R$ -matrix  $R$  (18)) is given by

$$v \otimes w \mapsto c(x, y)(\partial(x) \triangleright w \otimes v),$$

for homogeneous  $v, w$  of degrees  $x, y \in X$ . Comparing with Proposition 5.1 we obtain the stated assertion.  $\square$

We next explain the relation between the quasi-triangular quasi-Hopf algebras constructed above and the twisted Drinfeld double of a finite group. Let  $\omega$  be a normalized 3-cocycle on a finite group  $G$ .

Define

$$(21) \quad \gamma_{g,h}(x) := \frac{\omega(g, h, x)\omega(ghxh^{-1}g^{-1}, g, h)}{\omega(g, h, xh^{-1}, h)}$$

and

$$(22) \quad \mu_g(x, y) := \frac{\omega(gxg^{-1}, g, y)}{\omega(gxg^{-1}, gyg^{-1}, g)\omega(g, x, y)},$$

for all  $g, h, x, y \in G$ .

A direct computation establishes the following.

**Lemma 6.3.** *The quadruple  $(\omega, \gamma, \mu, 1)$  where  $\gamma$  and  $\mu$  are defined by (21) and (22), respectively, is a quasi-abelian 3-cocycle on the crossed module  $(G, G, \text{id}_G)$  (where  $G$  acts on itself by conjugation) in the sense of Definition 3.4.*

Let  $(\omega, \gamma, \mu, 1)$  be the quasi-abelian 3-cocycle on  $(G, G, \text{id}_G)$  constructed in Lemma 6.3. Then, evidently,  $H(\omega^{-1}, \gamma^{-1}, \mu^{-1}, 1) \cong D^\omega(G)$ , as quasi-triangular quasi-Hopf algebras. In particular,  $\mathcal{C}(\omega^{-1}, \gamma^{-1}, \mu^{-1}, 1)^G \cong D^\omega(G)\text{-Mod}$ , as braided fusion categories.

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DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TX 77843, USA.  
*E-mail address:* `dnaidu@math.tamu.edu`